

Modeling Viscoelastic Properties of Loudspeaker Suspensions Using Fractional Derivatives

ANTONIN NOVAK

(antonin.novak@univ-lemans.fr)

A lot of materials used in loudspeaker suspensions exhibit significant frequency dependence of damping and compliance due to their various viscoelastic properties. Few empirical models have been proposed to take into account these variations, the most used one is the LOG model proposed by Knudsen and Jensen. On the other hand, the fractional calculus has been successfully applied to characterize the rheological properties of viscoelastic materials including their frequency dependence. This study shows that the loudspeaker suspension can be successfully modeled using fractional derivatives. The traditional low-frequency loudspeaker model is extended by a simple fractional element. It is shown on four loudspeakers with different types of suspension with different properties that the proposed model fits well with all the studied cases.

0 INTRODUCTION

A traditional electrodynamic loudspeaker is equipped with a suspension that is usually formed by both a surround attached to the outer edge of the diaphragm and a spider attached to the voice-coil. The purpose of the suspension is to center and adjust the voice-coil in the air gap and to allow a desired axial motion of the diaphragm while preventing lateral motion or rocking. The stiffness of the surround, along with the moving mass, determines the resonance frequency of the loudspeaker and the cone excursion.

Materials used for loudspeaker suspensions have been improved dramatically over the last ten years and will probably keep improving within the next decades. While the spider is usually made of impregnated textiles (cotton, poly-cotton, nomex), the surround can be made of different types of materials such as rubber, foam, coated fabric or diaphragm material (rigid mounting). These materials exhibit various viscoelastic properties leading to different loudspeaker characteristics.

It has been shown in several studies [1–3] that the traditional model of loudspeaker fails to describe the complicated viscoelastic behavior of these materials. Thus, several different models have been proposed based either on rheological or on empirical approaches [1–3]. These models are discussed in Sec. 1.

On the other hand, many physical processes, including the viscoelastic materials, exhibit fractional order behavior [4]. The fractional derivatives, discussed in Sec. 2, provide an adequate description of real viscoelastic behavior with a limited number of material parameters [5]. Moreover, there

exists a physical interpretation of the fractional derivatives [6–9] giving them a full credit over the purely empirical models.

This paper shows how fractional derivatives can provide a suitable method of describing the properties of viscoelastic materials used for loudspeaker suspensions. The aim of the paper is not to evaluate a general theory of fractional derivatives, nor to provide an extensive comparison of viscoelastic models, but to show how to successfully use fractional derivatives for suspension modeling. Adding a single fractional viscoelastic element to the traditional model of loudspeaker, discussed in Sec. 3, leads to a very good agreement between the measured data and the model. Four loudspeakers, each of them equipped with different types of surround, are used to test the proposed fractional model. The loudspeakers, the experimental setup, and the results are discussed in Sec. 4. Moreover, the frequency dependence of the compliance and the mechanical losses is studied in Sec. 5 and a comparison between the measured data and the model based on fractional derivatives is provided. The proposed fractional model provides a very good fit with the measured data in all studied cases.

1 VISCOELASTIC MODELS OF SUSPENSION

The materials used for loudspeaker suspensions exhibit both viscous and elastic characteristics. The rheological models describing these characteristics consist of a combination of basic elements such as linear elastic springs (Fig. 1a) and viscous dampers (Fig. 1b) [10]. Simple

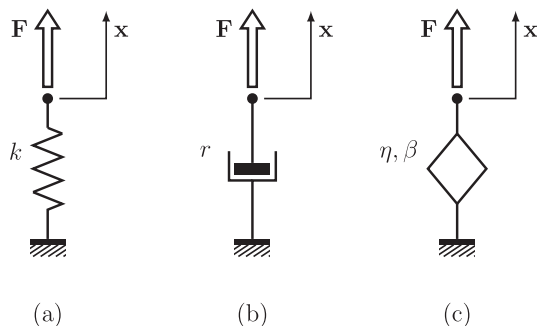


Fig. 1 Basic rheological elements: (a) a purely elastic element = spring, (b) a purely viscous element = damper, (c) a fractional element = interpolation between the spring and the damper.

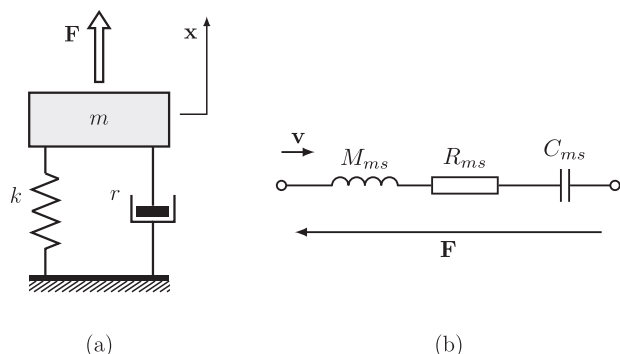


Fig. 2 (a) A traditional mass-spring-damper system based on Kelvin-Voigt viscoelastic model consisting of a Newtonian damper and a Hookean elastic spring connected in parallel. (b) Its electro-mechanical analogy.

models, such as Kelvin-Voigt and Maxwell ones, use one spring and one damper connected in parallel and in series respectively. More complicated models consist of more springs and dampers, such as a Standard Linear Solid model (SLS), also known as the Zener model, combining three elements [11]. Obviously, the greater the number of springs and dampers, the better the fitting the material behavior of the suspension. The most general viscoelastic models used in rheology, such as Generalized Kelvin-Voigt or the Generalized Maxwell ones, consist of an arbitrary number of elements [11–13].

1.1 Traditional Model

The traditional well-known loudspeaker model [14, 15] is based on a simple mass-spring-damper system (Fig. 2a). In such a model the mechanical part consists of a mass m attached to a parallel combination of a Newtonian damper defined by its viscous damping coefficient r and a Hookean elastic spring defined by its stiffness k . The electro-mechanical analogy to such a system is depicted in (Fig. 2b) in which the mass, damping coefficient (also called mechanical resistance), and compliance (the inverse of stiffness) are represented by symbols M_{ms} , R_{ms} , and C_{ms} as it is a common practice in the field of audio engineering. Such a model is widely used not only by audio engineers but

also by many scientists most likely due to its low number of parameters and its simplicity.

It is, however, commonly experienced that the traditional three parameter-model based on a simple mass-spring-damper does not describe correctly the real-world behavior of a viscoelastic material. One of the drawbacks of the traditional model is that its mechanical losses are represented purely by viscous damper, omitting possible elastic or viscoelastic losses. Another important drawback is that it does not account for the viscoelastic creep at low frequencies observed in many previous works dealing with loudspeaker suspensions [1, 16, 17]. A natural consequence of such drawbacks is an effort to improve the classical viscoelastic model in such a way that it fits the real-world behavior of a viscoelastic material correctly while still keeping the number of parameters low.

1.2 Linear Solid Models

A straightforward solution to improve the model is to add spring and damper elements to the Kelvin-Voigt model in series, parallel or in combination of both.

The Standard Linear Solid (SLS) model is a commonly used rheological model for viscoelastic materials. It ameliorates the Kelvin-Voigt model by putting a spring element in series. The study of Knudsen and Jensen [1], in which the spring of the traditional model is replaced by the SLS model, has shown that such a model improves the traditional model. Such a conclusion is not surprising as two extra parameters are added to the model.

More general linear solid models such as the Generalized Maxwell model [12], or parallel connection of so-called three-parameter model [13] may also better fit the material behavior of the suspension, yet at the cost of a greater number of parameters.

1.3 Models with Frequency Dependent Parameters

In the study of Knudsen and Jensen [1], two models containing only one extra parameter have been tested: the EXP (exponential) and the LOG (logarithmic) model. Both models use a complex frequency dependent compliance defined as $c_e(j\omega)^{-\beta}$ and $c_l[1 - \lambda \log(j\omega)]$ respectively. According to studies in which the LOG model is compared with the SLS model [17], the LOG model performs much better than the SLS model. A similar conclusion is also found in [1], where the LOG is performing the best.

The LOG model becomes a widely accepted model of complex compliance [2, 3] for its simplicity (only one extra parameter is needed) and for its sufficiently accurate prediction of the behavior of a viscoelastic material.

However, as the LOG model is purely empirical, and not derived from physical considerations [3], it involves two important drawbacks. First, as the frequency approaches zero, the compliance of the LOG model approaches infinity. Second, above a certain frequency the LOG model predicts negative compliance that is physically impossible [2, 3]. Moreover, according to [18], the LOG model performs well

for suspensions with a high degree of viscoelasticity but fails when the degree of viscoelasticity is low.

In the work of Thorgborg et al. [2], a modified LOG model is proposed including only frequency-dependent damping (FDD). The FDD model is refined in [18] to also include the frequency dependence of compliance (modified FDD). Contrary to the original LOG model, the frequency dependence of damping and the frequency dependence of compliance are separated, involving one extra parameter. Nevertheless, even if the separation of frequency dependencies of damping and compliance may get more a physical sense, the model remains still empirical and has the same drawbacks as the LOG model.

1.4 Models Based on Retardation Spectrum

Recently, Ritter and Agerkvist [3] found an interesting connection between the empirical LOG model and the Generalized Kelvin-Voigt model (a physical model with an infinite number of elements) through the retardation spectrum [10, 19]. Ritter and Agerkvist proposed two new models (3PC and 4PC). The 3PC model uses three parameters to describe the complex frequency dependent compliance and corrects the problem of negative compliance at higher frequencies. The 4PC model corrects both drawbacks of the LOG model by defining a minimum compliance and an equilibrium compliance. According to the authors, the 4PC model, which uses four parameters to model the complex compliance, provides slightly better accuracy than the LOG model.

2 FRACTIONAL VISCOELASTIC ELEMENT

The use of fractional derivatives is nowadays accepted in many fields of physics and engineering [4] including mechanics [20], acoustics [21, 22], and more recently audio engineering [23]. In [23] the fractional derivatives are successfully used to model the electrical impedance of a loudspeaker motor.

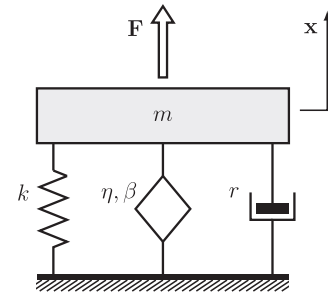
The purpose of this paper is not to give a review of fractional derivatives since much information can be found in [8, 24, 25] but to show that the fractional derivatives can describe better the complex viscoelastic behavior of the loudspeaker suspension, and, consequently, can be successfully used in modeling the loudspeaker mechanical part.

The fractional derivatives are based on the generalization of the differential operator $\frac{d^n}{dt^n}$, in which n is a positive integer number, to $\frac{d^\beta}{dt^\beta}$ where β is any real number. A typical application of fractional derivatives is a rheological *fractional element* depicted in Fig. 1c. The force-displacement relation of such an element is given by

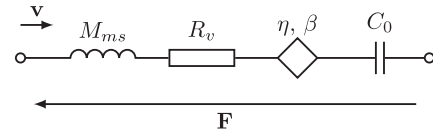
$$\mathbf{F}(t) = \eta \frac{d^\beta \mathbf{x}}{dt^\beta}, \quad (1)$$

where the constant η is called a viscoelastic coefficient, its physical dimension being $[\text{Ns}^\beta/\text{m}]$ [26].

Comparing the fractional element with the traditional elements such as spring (Fig. 1a) and damper (Fig. 1b), we conclude that relation Eq. (1) can represent the purely



(a)



(b)

Fig. 3 (a) A proposed mechanical model of the loudspeaker suspension in which a fractional element is added in parallel to the traditional mass-spring-damper system. (b) Its electro-mechanical analogy.

elastic element (spring) by putting $\beta = 0$ as well as the purely viscous element (damper) by putting $\beta = 1$. The fractional element can be seen as an interpolation between the spring and the damper; $\beta \in (0, 1)$ indicating the degree between the viscosity and the elasticity. Some authors [27] associate the coefficient $\beta \in (0, 0.5)$ to the elasto-viscous materials and the coefficient $\beta \in (0.5, 1)$ to the visco-elastic materials.

3 APPLICATION OF FRACTIONAL DERIVATIVES TO THE LOUDSPEAKER SUSPENSION

In this section the fractional derivatives are applied to the classical mechanical equation of a loudspeaker that describes the simple mass-spring-damper system of Fig. 2 by

$$\mathbf{F}(t) = M_{ms} \frac{d^2 \mathbf{x}}{dt^2} + R_{ms} \frac{d\mathbf{x}}{dt} + \frac{1}{C_{ms}} \mathbf{x}, \quad (2)$$

$\mathbf{F}(t)$ being the force applied to the mechanical part of the loudspeaker and $\mathbf{x}(t)$ being its consecutive displacement.

Adding the fractional element depicted in Fig. 1c in parallel to the traditional model of Fig. 2a, we obtain a new proposed model depicted in Fig. 3. The relation between the force and the displacement can be defined in the time-domain as

$$\mathbf{F}(t) = M_{ms} \frac{d^2 \mathbf{x}}{dt^2} + R_v \frac{d\mathbf{x}}{dt} + \eta \frac{d^\beta \mathbf{x}}{dt^\beta} + \frac{1}{C_0} \mathbf{x}, \quad (3)$$

or in the frequency domain, defining $F(\omega)$ and $X(\omega)$ as the Fourier transforms of $\mathbf{F}(t)$ and $\mathbf{x}(t)$, as

$$\frac{F(\omega)}{X(\omega)} = -\omega^2 M_{ms} + j\omega R_v + (j\omega)^\beta \eta + \frac{1}{C_0}. \quad (4)$$

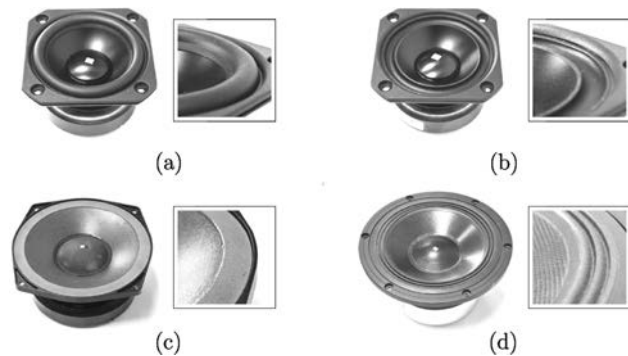


Fig. 4 Evaluated loudspeakers: (a) Peerless PLS-75F25AL02-08, (b) Peerless PLS-75F25AL04-08, (c) PHL 1520, (d) Pioneer W16FU90-51D-T.

In this modified model R_v represents viscous losses and C_0 represents the static compliance. The mechanical impedance $Z_m(\omega)$, defined as the force over the velocity, of such a system is then defined as

$$Z_m(\omega) = j\omega M_{ms} + R_v + (j\omega)^{(\beta-1)}\eta + \frac{1}{j\omega C_0}. \quad (5)$$

4 EXPERIMENTS

In the following, four different loudspeakers are evaluated, each of them having a different type of suspension.

Peerless PLS-75F25AL02-08 is a 3" compact full range driver equipped with an anodized aluminium cone, a large roll SBR low-damping rubber surround, and a Nomex spider (Fig. 4a).

Peerless PLS-75F25AL04-08 is an almost identical copy of the first loudspeaker (3" full range driver) except that this one is equipped with an inverted rubber surround (Fig. 4b).

PHL 1520 is a 6.5" midrange driver with a high-strength cellulose fiber cone impregnated and coated on both sides with damped resins. The suspension consists of a cup spider made of Nomex and a high-speed flat damped surround made of a light-weight strip of polyethylene foam (Fig. 4c).

Pioneer W16FU90-51D-T is a 6.5" woofer with a woven aramid/carbon composite shell cone suspended by a corrugated surround and a cup spider made of Nomex (Fig. 4d).

4.1 Measurement Set-Up

A block diagram of the measuring system used to both excite the loudspeaker under test and acquire the data is shown in Fig. 5. The loudspeaker is driven with a voltage source (a source having a very low output impedance) as it is usually operated in actual conditions. An NI-4431 USB acquisition device is used to send and acquire the measured signals and a Devialet D-Premier amplifier to drive the loudspeaker. The current flowing through the loudspeaker is measured using a Current Shunt (N4L HF003) with a resistance of $0.48 \sim \Omega$. The displacement is measured using a Polytec OFV-3000 vibrometer.

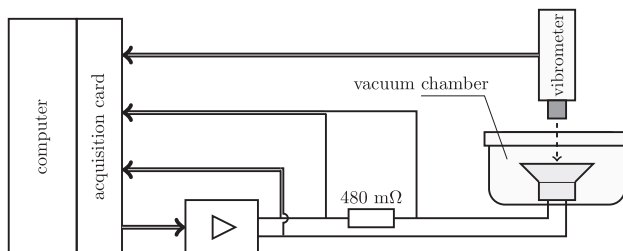


Fig. 5 A block diagram of the measuring system. The measurements are performed for two different configurations: under vacuum of -0.97 bar (the vacuum chamber is closed), and in standard atmosphere (the vacuum chamber is opened).

Prior to measurements, the loudspeakers are run-in for several hours using a low-frequency high-displacement signal to ensure mechanical stability during the measurement. Each loudspeaker is put into a small vacuum chamber (Klippel Vacuum Kit) and two series of measurements are made: in standard atmosphere (vacuum chamber opened) and under vacuum of -0.97 bar. In vacuum, no air should influence the motion of mechanical part of the loudspeaker, possibly responsible for a fraction of mechanical losses caused by the viscosity of the air in the gap [28, 29]. Also, the radiation impedance should be zero. Since the radiation impedance in standard atmosphere and at low frequencies is dominated by a mass loading reactance, it is equivalent to adding a mass proportional to the equivalent surface of the membrane. Thus, the reason to measure the loudspeaker in vacuum and in standard atmosphere is twofold: first to test if the proposed model estimates a similar difference in mass for same size of membranes and, second, to verify if the presence of air influences the mechanical losses.

4.2 Measurement Conditions

The mechanical properties of the loudspeaker may vary with time due to mechanical fatigue [30, 31], temperature or humidity variation. Therefore, rather than using a sine measurement in which each frequency is applied at a different time, it is preferable to use a multi-tone stimuli where all the frequencies are applied at the same time. The loudspeaker is driven by a multi-tone signal with 80 frequencies logarithmically distributed between 0.1 Hz and 1 kHz. The exact value of each frequency component is adjusted to be a divisor of the sampling frequency ($f_s = 5$ kHz) of the acquisition system. The phase of each frequency is chosen arbitrarily to avoid a huge crest factor of the input signal.

The choice of the input level is affected by two contradictory facts: it must not be too low, otherwise the signal would be drawn in the noise, but also not too high to overcome the problems associated with nonlinearities of the loudspeaker. A signal with a peak value not exceeding 0.2 V is chosen as a compromise.

The acquired voltage $u(t)$, current $i(t)$, and displacement $x(t)$ are recorded, and the Fourier coefficients of each frequency component $\omega_i = 2\pi f_i$ of each signal are calculated, resulting in $U(\omega_i)$, $I(\omega_i)$, and $X(\omega_i)$. Approaching the frequency 300 Hz, the assumption of the piston motion of the

membrane is not valid due to cone breakup-rim resonances. Since the measured displacement $x(t)$ of the cone is not reliable above this frequency, the measured data above 300 Hz are ignored.

4.3 Results

Force factor Bl is first estimated from measured data [32] and the force $F(\omega)$ applied on the mechanical part is calculated as $F(\omega) = Bl \cdot I(\omega)$. The receptance frequency response functions [33] $X(\omega)/F(\omega)$ are next derived as well as the mechanical impedance $Z_m(\omega) = F(\omega)/V(\omega)$.

To estimate the best set of parameters of the fractional model a curve fitting procedure is performed by minimizing the sum of error (least-squares fit) between 0.1 Hz and 300 Hz. The fit is performed on both the real and the imaginary part of the mechanical impedance Eq. (5), expressed as

$$\Re\{Z_m\} = R_v + \sin\left(\beta\frac{\pi}{2}\right)\omega^{(\beta-1)}\eta, \quad (6)$$

and

$$\Im\{Z_m\} = \omega M_{ms} - \cos\left(\beta\frac{\pi}{2}\right)\omega^{(\beta-1)}\eta - \frac{1}{\omega C_0}. \quad (7)$$

The estimated parameters are listed in Table 1 for the four measured loudspeakers and for both configurations: measured in standard atmosphere, noted as (atm), and measured in vacuum. The respective RMS errors (magnitude and phase) of receptances $X(\omega)/F(\omega)$ are listed in the same Table. The most important conclusions from the estimated parameters are the following.

(1) The RMS errors are kept very low for both magnitude and phase for all measured loudspeakers. The error of magnitude is between 0.22% and 0.55% and the phase error between 0.14 and 0.44 degrees. A similar conclusion can be drawn from Fig. 6 in which the receptances $X(\omega)/F(\omega)$ obtained from measurements in vacuum (dots) are compared to simulated receptances calculated using the model with fractional derivatives. The simulated curves almost perfectly coincide with the measured data, even for very low frequencies in which a so-called creep phenomenon appears.

(2) The estimated mass M_{ms} is always lower for measurements in a vacuum, which agrees with physical laws.

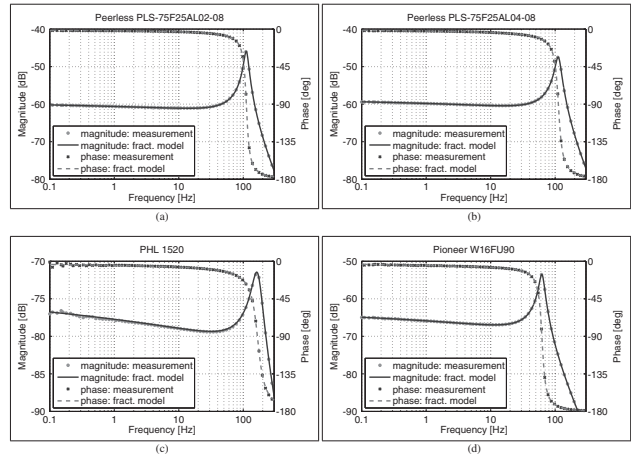


Fig. 6 Measured receptance $X(\omega)/F(\omega)$ of four different loudspeakers (dots) compared to a simulation calculated using the model with fractional derivatives. Measurements were performed under vacuum of -0.97 bar.

Moreover, the mass difference between the measurements in a standard atmosphere and in a vacuum, corresponding to the air load mass, is 0.16 g and 0.15 g in the case of the 3" loudspeakers and 1.9 g and 2.2 g for 6.5" speakers. The two same-size pairs give almost the same air load mass. Considering the size of equivalent surface of each loudspeaker and their approximative theoretical values of air load [34], the indirectly estimated values of air load mass are reliable.

(3) The estimated mechanical parameters C_0 , R_v , η , and β defining the viscoelastic properties of the loudspeaker are almost independent on the atmospheric condition (standard atmosphere or vacuum) for both Peerless speakers and for the PHL speaker. The static compliance C_0 is almost the same for both Peerless speakers as well as for the Pioneer speaker. A 10% difference between the measurement of the static compliance C_0 in a standard atmosphere and a vacuum is found in the case of the PHL speaker with the foam surround. The viscoelastic parameters η and β do not vary a lot with the atmospheric condition (standard atmosphere or vacuum). The fractional order β always drops a little when changing the atmospheric conditions from vacuum to standard atmosphere but the variation remains very small

Table 1 Estimated mechanical parameters of the model with fractional derivatives for all four loudspeakers measured in standard atmosphere (atm) and in vacuum and their respective RMS errors (magnitude and phase) of responses from force to displacement $X(\omega)/F(\omega)$.

		parameters					RMS error of $X(\omega)/F(\omega)$	
		M_{ms} [g]	C_0 [mm/N]	R_v [Ns/m]	η [Ns $^\beta$ /m]	β [-]	mag. [%]	\angle [deg]
Peerless PLS-75F25AL02-08	atm	2.69	1.12	0.167	140	0.140	0.22	0.13
	vacuum	2.53	1.12	0.173	140	0.141	0.26	0.17
Peerless PLS-75F25AL04-08	atm	2.52	1.20	0.182	111	0.173	0.25	0.19
	vacuum	2.37	1.18	0.190	96.6	0.194	0.31	0.21
PHL 1520	atm	11.9	0.182	1.95	1889	0.146	0.51	0.38
	vacuum	10.0	0.203	2.26	2141	0.150	0.55	0.44
Pioneer W16FU90	atm	19.4	0.740	0.754	402	0.170	0.31	0.23
	vacuum	17.2	0.722	0.236	424	0.189	0.27	0.16

($\approx 10\%$). The most sensitive parameter to the variation of atmospheric condition is the viscous coefficient R_v . The viscous coefficient R_v of the Pioneer speaker drops by a factor of 3 when being in a vacuum suggesting possible losses due to viscosity of the air in the gap, but obviously more scientific proofs would be required to confirm or decline such a conclusion. On the other hand the viscous coefficient R_v increases by 4% for both Peerless speakers and by 15% for the PHL speaker when being in a vacuum. The possible explanation for this variation may be simple mechanical fatigue [30, 31] or a temperature variation influencing the viscoelastic properties of suspension materials. Being aware of a possible decrease of temperature in a vacuum, it is measured using a liquid crystal thermometer strip. The drop of temperature is approximately $4 \pm 0.5^\circ\text{C}$; from $20 \pm 0.5^\circ\text{C}$ in standard atmosphere to $16 \pm 0.5^\circ\text{C}$ in vacuum.

(4) Finally, viscoelastic coefficient β remains low for all four tested loudspeakers, corresponding to rather elasto-viscous than visco-elastic behavior [27]. It is also interesting to note that the coefficient β is different for both Peerless loudspeakers. These loudspeakers use the same spider and the surround is made of the same rubber material; the only difference is the shape of the surround (a normal surround and an inverted one). The results suggest that the shape of the surround may also influence the model parameters.

5 FREQUENCY DEPENDENT COMPLIANCE AND LOSSES

It has been shown in many papers dealing with viscoelastic behavior of loudspeaker suspensions [1, 3, 18] that the compliance as well as the mechanical losses are frequency dependent. The frequency dependent complex compliance [1] can be written as $C^*(\omega) = C'(\omega) - jC''(\omega)$ [10], where the real-part component $C'(\omega)$ is called the *storage compliance* and determines the storage energy, while the imaginary-part component $C''(\omega)$ is responsible for energy losses and is consequently called *loss compliance*.

5.1 Storage Compliance

The complex compliance [10, 35] of a mechanical system can be expressed as [3]

$$C^*(\omega) = \left[\frac{F(\omega)}{X(\omega)} + \omega^2 M_{ms} \right]^{-1} \quad (8)$$

where all the viscous and viscoelastic losses are included in the imaginary part of the complex compliance $C^*(\omega)$.

The complex compliance of the proposed fractional model can be thus derived from Eq. (4) and Eq. (8) as

$$C^*(\omega) = \left[\frac{1}{C_0} + (j\omega)^\beta \eta + j\omega R_v \right]^{-1}. \quad (9)$$

Studying closely the storage compliance $C'(\omega)$, defined as a real part of Eq. (9), we can note that the fractional order model does not suffer from the drawbacks of the LOG model discussed in Sec. 1.

First, as the frequency approaches zero, the storage compliance approaches the static compliance C_0 . Second, con-

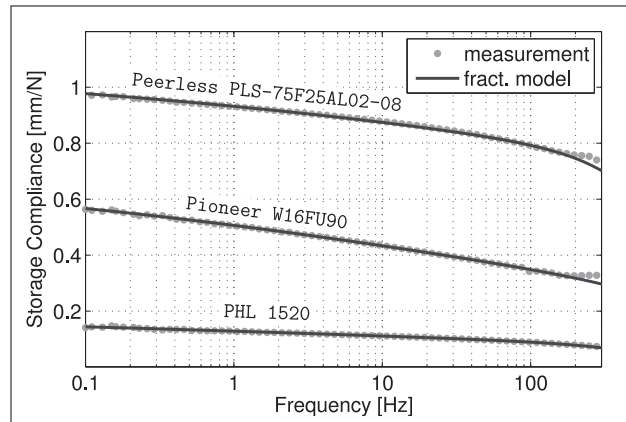


Fig. 7 Storage compliance of three different loudspeakers derived from the measurement (dots) compared to a simulation calculated using the model with fractional derivatives. Measurements were performed under vacuum of -0.97 bar.

trary to the LOG model, the storage compliance never reaches negative values. In Fig. 7 the storage compliance derived from measured data in a vacuum using Eq. (8) is compared with the simulated storage compliance calculated using the fractional model Eq. (9). The comparison in Fig. 7 is provided for all the tested loudspeakers except the Peerless with inverted surround since its estimated mechanical parameters, and consequently the storage compliance, are very similar to the Peerless loudspeaker with normal surround and the very close curves would worsen the readability of the graph. We can conclude from Fig. 7 that a good estimation of all the parameters of the proposed fractional model is sufficient to describe correctly the frequency dependent storage compliance within the frequency range of interest.

When comparing the behavior of the storage compliance (Fig. 7) at very low frequencies, we can see that the measured data as well as the fractional model approaches the static compliance C_0 (see Table 1) as frequency tends to zero. However, all tested loudspeakers still exhibit an increase of storage compliance with decreasing frequency even for frequencies approaching 0.1 Hz. Therefore, it still remains unclear at which (very low) frequency the compliance stabilizes to the value of static compliance C_0 .

5.2 Mechanical Losses

When studying the mechanical properties of a loudspeaker, the mechanical losses are usually defined as the real part of the mechanical impedance. In a traditional model of loudspeaker Eq. (2), the only mechanical losses taken into account are the viscous ones, represented by the viscous coefficient R_{ms} . The mechanical losses of the proposed fractional model have two components Eq. (6): a constant R_v responsible for the viscous losses and a component with frequency dependence $\omega^{(\beta-1)}$ representing the losses caused by the viscoelastic fractional element.

In Fig. 8 the mechanical losses derived from measured data in a vacuum as a real part of the measured mechanical impedance $Z_m(\omega)$ are compared with the mechanical losses

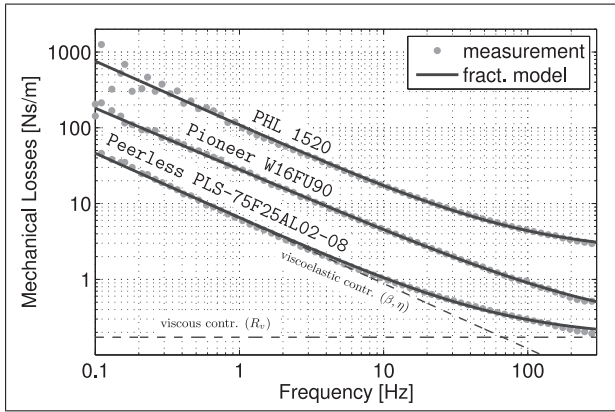


Fig. 8 Mechanical losses of three different loudspeakers derived from the measurement (dots) compared to a simulation calculated using the model with fractional derivatives. Measurements were performed under vacuum of -0.97 bar. The relative contribution of the viscoelastic losses (β , η) and viscous losses (R_v) of the Peerless loudspeaker are depicted as dashed asymptotes.

calculated using the fractional model Eq. (6). Once again, we can conclude from Fig. 8 that the fractional model gives a very good fit between the measured data and the data calculated using the fractional model.

The two dashed asymptotic lines in Fig. 8 show the relative contribution of the viscous losses (parameter R_v) and viscoelastic losses (parameters β and η) for the Pioneer speaker. Note, that the viscous contribution (R_v), usually used as a single parameter to represent losses in the traditional model, is not frequency dependent and can hardly fit the measured decreasing curve. The viscoelastic contribution is represented by a straight line in a log-log graph; the slope being given by $(\beta - 1)$.

5.3 Loss Factor

It is sometimes convenient to define a loss factor $\tan \delta$ [35] relating the storage and the loss energy by the following relation

$$C^*(\omega) = C'(\omega)[1 - j \tan \delta(\omega)], \quad (10)$$

where

$$\tan \delta = \frac{C''(\omega)}{C'(\omega)}. \quad (11)$$

The loss factor of the fractional model can be thus derived from Eq. (9) and Eq. (11) as

$$\tan \delta = \frac{\omega R_v + \eta \sin(\beta \frac{\pi}{2}) \omega^\beta}{\frac{1}{C_0} + \eta \cos(\beta \frac{\pi}{2}) \omega^\beta}. \quad (12)$$

In Fig. 9 the loss factors derived from measured data in a vacuum Eq. (11) are compared with the loss factors calculated using the fractional model Eq. (12). The asymptotic behavior due to viscous and viscoelastic contributions are depicted using dashed lines. Note that the slope of the dashed line of the viscous contribution in the log-log plot is equal to one no matter the coefficients of the model ($\tan \delta = R_v C_0 \omega$ for $\eta = 0$). Thus, considering only viscous contri-

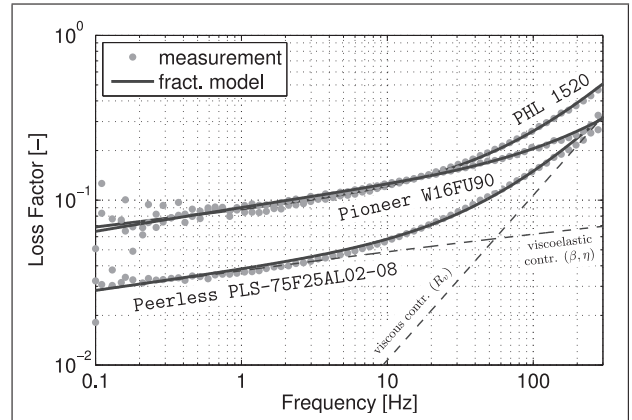


Fig. 9 Loss factor $\tan \delta$ of three different loudspeakers derived from the measurement (dots) compared to a simulation calculated using the model with fractional derivatives. Measurements were performed under vacuum of -0.97 bar. The relative contribution of the viscoelastic losses (β , η) and viscous losses (R_v) of the Peerless loudspeaker are depicted as dashed asymptotes.

bution as a single element responsible for the losses would not be sufficient especially at low frequencies where the slope of the log-log curve of $\tan \delta$ is much lower than one.

6 DISCUSSION

6.1 Frequency Range

The results of four studied loudspeakers with different types of suspension show that the proposed model with a single fractional element provides very low rms error within the frequency range in which the measured data are reliable. Since the assumption of the piston motion of the membrane is not valid at higher frequencies due to cone breakup-rim resonances, usually occurring two or three octaves above the resonance frequency, the model accuracy cannot be estimated above these frequencies. However, as shown in Figs. 8 and 9 the viscoelastic contribution of the model is much more important under the resonance frequency. Consequently, the frequency limit of the model due to the cone breakup-rim resonances is the same as for the traditional lumped model of an electrodynamic loudspeaker.

6.2 Number of Parameters

The proposed model uses four parameters to describe the viscoelastic behavior of the mechanical part: the static compliance C_0 defining the purely elastic properties of the suspension, the viscous coefficient R_v defining purely viscous properties, the viscoelastic coefficient η , and the fractional order β . More complicated models such as Kelvin-Voigt chain model [36], or Maxwell chain model [36–38] use more fractional elements to fit better the viscoelastic behavior of the material but at the cost of greater number of parameters. Lower number of parameters would also be possible but at the cost of the model precision. Keeping only the fractional element with the spring (putting $R_v = 0$) would result in a straight line of the log-log plot of mechanical losses Eq. (6) in Fig. 8. Such a choice would lead

to an incorrect fit of mechanical losses and consequently to higher error of the model. Keeping only the fractional element with the damper (putting $C_0 = \infty$) would bring the same drawback as the one with the LOG model. Moreover, such a solution has already been proposed by Knudsen and Jensen [1], since the EXP model is indeed a particular case of the proposed fractional model with $C_0 = \infty$.

6.3 Nonlinear Fractional Model

The proposed fractional model can be applied in the nonlinear domain in several manners. A simple engineering approach would be to make vary at least one of the model parameters R_v , η , β , and C_0 with displacement. A more general approach would consist in using a nonlinear model such as Volterra model [39] or Generalized Hammerstein model [40] in which the fractional derivatives would appear inside the nonlinear kernels.

The linear relation between the force and the displacement Eq. (3) can be expressed as

$$\mathbf{F}(t) - M_{ms} \frac{d^2 \mathbf{x}}{dt^2} = \hat{\mathbf{S}}[\mathbf{x}(t)], \quad (13)$$

where $\hat{\mathbf{S}}$ represents a linear operator defined as

$$\hat{\mathbf{S}}[\mathbf{x}(t)] = \left[R_v \frac{d}{dt} + \eta \frac{d^\beta}{dt^\beta} + \frac{1}{C_0} \right] \mathbf{x}. \quad (14)$$

Since the operator \mathbf{S} is linear and translation invariant, it can be represented as a convolution

$$\hat{\mathbf{S}}[\mathbf{x}(t)] = \mathbf{h}(t) * \mathbf{x}(t), \quad (15)$$

where the kernel $\mathbf{h}(t)$ is defined using inverse Fourier transform \mathcal{F}^{-1} as

$$\mathbf{h}(t) = \mathcal{F}^{-1} \left[j\omega R_v + \eta(j\omega)^\beta + \frac{1}{C_0} \right]. \quad (16)$$

In a case of a nonlinear system, the linear operator $\hat{\mathbf{S}}$ must be replaced by a nonlinear operator $\hat{\mathbf{S}}_{NL}$. Using the Generalized Hammerstein model, the $\hat{\mathbf{S}}_{NL}$ expands to

$$\hat{\mathbf{S}}_{NL}[\mathbf{x}] = \sum_{n=1}^N \mathbf{h}_n(t) * \mathbf{x}^n(t). \quad (17)$$

The fractional derivatives may thus apply not only to the linear kernel $\mathbf{h}_1(t)$ Eq. (16) but to all nonlinear kernels $\mathbf{h}_n(t)$.

Moreover, taking into account a possible hysteresis-like dependence on the peak value of displacement x_{max} [41], the nonlinear operator $\hat{\mathbf{S}}_{NL}$ should be implemented as a function of instantaneous and peak displacement leading to much more complicated models [42] that require advanced identification techniques of nonlinear systems [43].

7 CONCLUSION

This paper shows that the mechanical models based on fractional derivatives may prove useful for loudspeaker suspensions. In view of their ability to model viscoelastic phenomena, they provide a suitable method of describing dynamical properties of viscoelastic materials used in loudspeaker manufacture.

The results of four studied loudspeakers with different types of suspension show that the proposed model with a single fractional element provides very low rms error when compared to the measured data for all loudspeakers measured in a standard atmosphere as well as in a vacuum.

8 REFERENCES

- [1] M. H. Knudsen and J. G. Jensen, "Low-Frequency Loudspeaker Models that Include Suspension Creep," *J. Audio Eng. Soc.*, vol. 41, pp. 3–18 (1993 Jan./Feb.).
- [2] K. Thorborg, C. Tinggaard, F. Agerkvist, and C. Futtrup, "Frequency Dependence of Damping and Compliance in Loudspeaker Suspensions," *J. Audio Eng. Soc.*, vol. 58, pp. 472–486 (2010 Jun.).
- [3] T. Ritter and F. T. Agerkvist, "Modeling Viscoelasticity of Loudspeaker Suspensions Using Retardation Spectra," presented at the *129th Convention of the Audio Engineering Society* (2010 Nov.), convention paper 8217.
- [4] J. Sabatier, O. P. Agrawal, and J. T. Machado, *Advances in Fractional Calculus*, vol. 4 (Springer, 2007).
- [5] N. Heymans and J.-C. Bauwens, "Fractal Rheological Models and Fractional Differential Equations for Viscoelastic Behavior," *Rheologica Acta*, vol. 33, no. 3, pp. 210–219 (1994).
- [6] H. Schiessel and A. Blumen, "Hierarchical Analogues to Fractional Relaxation Equations," *J. Physics A: Mathematical and General*, vol. 26, no. 19, pp. 5057–5069 (1993).
- [7] H. Schiessel, R. Metzler, A. Blumen, and T. Nonnenmacher, "Generalized Viscoelastic Models: Their Fractional Equations with Solutions," *J. Physics A: Mathematical and General*, vol. 28, no. 23, pp. 6567–6584 (1995).
- [8] R. Hilfer, P. Butzer, U. Westphal, J. Douglas, W. Schneider, G. Zaslavsky, T. Nonnenmacher, A. Blumen, and B. West, *Applications of Fractional Calculus in Physics*, vol. 128 (World Scientific, 2000).
- [9] K. D. Papoulia, V. P. Panoskaltsis, N. V. Kurup, and I. Korovajchuk, "Rheological Representation of Fractional Order Viscoelastic Material Models," *Rheologica Acta*, vol. 49, no. 4, pp. 381–400 (2010).
- [10] J. Ferry, *Viscoelastic Properties of Polymers* (Wiley, 1980).
- [11] G. E. Mase, *Schaum's Outline of Theory and Problems of Continuum Mechanics* (McGraw-Hill, New York, 1970).
- [12] A. Petosic, I. Djurek, and D. Djurek, "Modeling of an Electrodynamical Loudspeaker Including Membrane Viscoelasticity," presented at the *124th Convention of the Audio Engineering Society* (2008 May), convention paper 7356.
- [13] I. Djurek, A. Petosic, and D. Djurek, "Analysis of Viscoelasticity and Residual Strains in an Electrodynamical Loudspeaker," presented at the *126th Convention of the Audio Engineering Society* (2009 May), convention paper 7680.
- [14] N. Thiele, "Loudspeakers in Vented Boxes: Part 1," *J. Audio Eng. Soc.*, vol. 19, pp. 382–392 (1971 May).

- [15] R. H. Small, "Vented-Box Loudspeaker Systems—Part 2: Large-Signal Analysis," *J. Audio Eng. Soc.*, vol. 21, pp. 438–444 (1973 June).
- [16] M. H. Knudsen, P. Hansen, and J. G. Jensen, "The Significance of Viscoelastic Effects in Loudspeakers Parameter Measurements," presented at the *88th Convention of the Audio Engineering Society* (1990 Mar.), convention paper 2905.
- [17] F. Agerkvist, K. Thorborg, and C. Tinggard, "A Study of the Creep Effect in Loudspeaker Suspension," presented at the *125th Convention of the Audio Engineering Society* (2008 Oct.), convention paper 7561.
- [18] K. Thorborg and C. Futtrup, "Frequency Dependence of the Loudspeaker Suspension (A Follow Up)," *J. Audio Eng. Soc.*, vol. 61, pp. 778–786 (2013 Oct.).
- [19] M. Jirásek and P. Havlásek, "Accurate Approximations of Concrete Creep Compliance Functions Based on Continuous Retardation Spectra," *Computers & Structures*, vol. 135, pp. 155–168 (2014).
- [20] R. Koeller, "A Theory Relating Creep and Relaxation for Linear Materials with Memory," *J. Applied Mechanics*, vol. 77, no. 3, pp. 031008 (2010).
- [21] S. Holm and R. Sinkus, "A Unifying Fractional Wave Equation for Compressional and Shear Waves," *J. Acous. Soc. Am.*, vol. 127, no. 1, pp. 542–548 (2010).
- [22] S. Holm and S. P. Näsholm, "A Causal and Fractional All-Frequency Wave Equation for Lossy Media," *J. Acous. Soc. Am.*, vol. 130, no. 4, pp. 2195–2202 (2011).
- [23] P. Brunet and B. Shafai, "Identification of Loudspeakers Using Fractional Derivatives," *J. Audio Eng. Soc.*, vol. 62, pp. 505–515 (2014 Jul./Aug.).
- [24] I. Podlubny, *Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*, vol. 198 (Academic Press, 1998).
- [25] A. B. Malinowska, T. Odziejewicz, and D. F. Torres, *Advanced Methods in the Fractional Calculus of Variations* (Springer, 2015).
- [26] W. Grzesikiewicz, A. Wakulicz, and A. Zbiciak, "Non-Linear Problems of Fractional Calculus in Modeling of Mechanical Systems," *Int'l J. Mech. Sciences*, vol. 70, pp. 90–98 (2013).
- [27] M. Di Paola and M. Zingales, "Exact Mechanical Models of Fractional Hereditary Materials," *J. Rheol.*, vol. 56, no. 5, pp. 983–1004 (2012).
- [28] H. Kuttruff, *Acoustics: An Introduction* (CRC Press, 2007).
- [29] A. Voishvillo, "Identification Compression Driver Parameters Based on a Concept of Diaphragms Frequency-Dependent Area," presented at the *137th Convention of the Audio Engineering Society* (2014 Oct.), convention paper 9164.
- [30] F. Agerkvist and B. R. Pedersen, "Time Variance of the Suspension Nonlinearity," presented at the *125th Convention of the Audio Engineering Society* (2008 Oct.), convention paper 7560.
- [31] W. Klippel, "Mechanical Fatigue and Load-Induced Aging of Loudspeaker Suspension," presented at the *131st Convention of the Audio Engineering Society* (2011 Oct.), convention paper 8474.
- [32] A. Novak, P. Lotton, and L. Simon, "Nonlinear Force Factor Measurement of an Electrodynamic Loudspeaker," in *Forum Acusticum 2014* (2014 Sep.).
- [33] K. G. McConnell, *Vibration Testing: Theory and Practice* (John Wiley & Sons, 1995).
- [34] J. Vanderkooy, "The Inertial Air Load of a Loudspeaker Diaphragm," presented at the *124th Convention of the Audio Engineering Society* (2008 May), convention paper 7317.
- [35] M. Rousseau and J. Vanderkooy, "Visco-Elastic Aspects of Loudspeaker Drivers," presented at the *118th Convention of the Audio Engineering Society* (2005 May), convention paper 6524.
- [36] R. Koeller, "Applications of Fractional Calculus to the Theory of Viscoelasticity," *J. Applied Mechanics*, vol. 51, no. 2, pp. 299–307 (1984).
- [37] A. Hernandez-Jimenez, J. Hernandez-Santiago, A. Macias-Garcia, and J. Sanchez-Gonzalez, "Relaxation Modulus in PMMA and PTFE Fitting by Fractional Maxwell Model," *Polymer Testing*, vol. 21, no. 3, pp. 325–331 (2002).
- [38] Y. Yin and K.-Q. Zhu, "Oscillating Flow of a Viscoelastic Fluid in a Pipe with the Fractional Maxwell Model," *Applied Mathematics and Computation*, vol. 173, no. 1, pp. 231–242 (2006).
- [39] M. Schetzen, *The Volterra and Wiener Theories of Nonlinear Systems* (John Wiley & Sons, New York, 1980).
- [40] A. Novak, L. Simon, F. Kadlec, and P. Lotton, "Nonlinear System Identification Using Exponential Swept-Sine Signal," *Instrumentation and Measurement, IEEE Transactions on*, vol. 59, no. 8, pp. 2220–2229 (2010).
- [41] A. Novak, P. Lotton, and L. Simon, "Dynamic Measurement of Loudspeaker Suspension Parameters Using an Active Harmonic Control Technique," presented at the *136th Convention of the Audio Engineering Society* (2014 Apr.), convention paper 9054.
- [42] B. Maillou, "Identification et caractérisation des non-linéarités de systèmes acoustiques," Ph.D. thesis, LAUM, University of Maine, France (2015).
- [43] A. Novak, B. Maillou, P. Lotton, and L. Simon, "Nonparametric Identification of Nonlinear Systems in Series," *Instrumentation and Measurement, IEEE Transactions on*, vol. 63, no. 8, pp. 2044–2051 (2014).

THE AUTHOR



Antonin Novak

Antonin Novak was born in Prague, Czech Republic, in 1981. He received the Master degree in radio engineering from the Czech Technical University in Prague in 2006. In 2009, he received a Ph.D. degree in acoustics from the Czech Technical University in Prague and from Université du Maine, Le Mans, France, within the doctoral study under joint supervision. From 2009 he worked as a post-doctoral research fellow at the Laboratoire d'Acoustique

de l'Université du Maine, France (LAUM, UMR 6613). From 2011 to 2014 he worked for Orkidia Audio, France, where he was engaged in the research and development of innovative loudspeaker technology. He is currently a research fellow at Université du Maine, Le Mans, France. His research interests include analysis and measurement of non-linear systems, electroacoustics, and digital signal processing.